4.7 - Cauchy-Euler Equations

Definition: A **Cauchy-Euler equation** is a differential equation of the form $a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y = g(x)$. Earlier in the chapter we found solutions of the form $y = e^{mx}$. Here we will find solutions of the form $y = x^m$.

Solve the given differential equation.

Example: $x^2 y'' + 3xy' - 4y = 0$

$y = x^m \Rightarrow y' = m x^{m-1} \Rightarrow y'' = m(m-1) x^{m-2}$

$m(m-1) x^m + 3m x^m - 4x^m = 0$

$\Rightarrow m(m-1) + 3m - 4 = 0$

$m^2 + 2m - 4 = 0$

$m^2 + 2m + 1 = 4 + 1 \Rightarrow (m+1)^2 = 5$

$m = -1 \pm \sqrt{5}$

$$y = c_1 x^{-1+\sqrt{5}} + c_2 x^{-1-\sqrt{5}}$$

In general, for $ax^2y'' + bxy' + cy = 0$, the auxiliary equation is

$$am(m-1) + bm + c = 0$$

There are 3 possibilities:

Case I: $m_1 \neq m_2 \in \mathbb{R}$

$$\boxed{y = c_1 x^{m_1} + c_2 x^{m_2}}$$

Case II: $m_1 = m_2 = m$ (repeated)

$$\boxed{y = c_1 x^m + c_2 x^m \ln x} \leftarrow \ln^2 x, \ln^3 x, \ldots$$

Case III: $m_1 = \alpha + \beta i$, $m_2 = \alpha - \beta i$

$$\boxed{y = x^{\alpha}\left(c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)\right)}$$

Nonhomogeneous case

$ax^2 y'' + bxy' + cy = g(x)$

$\hookrightarrow$ std form $y'' + P(x)y' + Q(x)y = f(x)$

$\hookrightarrow$ variation of parameters

Example: $4x^2y'' + y = 0$

Example: $x^2y'' - 7xy' + 41y = 0$

$y = x^m \Rightarrow y' = mx^{m-1} \Rightarrow y'' = m(m-1)x^{m-2}$

$m(m-1) - 7m + 41 = 0$

$m^2 - 8m + 16 = -41 + 16$

$m = 4 \pm 5i$

$y = x^4(c_1 \cos(5\ln x) + c_2 \sin(5\ln x))$

A Cauchy-Euler equation can be converted to an equation with constant coefficients using the substitution $x = e^t$ and the Chain Rule.

$t = \ln x$

Example: Use the substitution $x = e^t$ to transform the given Cauchy-Euler equation to a differential equation with constant coefficients. Solve the original equation by solving the new equation using techniques previously learned.

$x^2y'' - 9xy' + 25y = 0$

$y = x^m$; $x = e^t \Rightarrow t = \ln x$

$y(x)$ becomes $y(t(x))$

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}\left(\frac{1}{x}\right)$$

$$\frac{d^2y}{dx^2} = -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dt}\right)$$

$$= \frac{1}{x^2}\left(-\frac{dy}{dt} + \frac{d^2y}{dt^2}\right)$$